Statistical Orbit Determination

Lecture 1 – Orbit Determination Concepts

Presenter: Christopher R. Simpson

Introduction

- Welcome!
  - Class Schedule: MWF
  - Non-traditional syllabus available here
- Working through material
  - If you have a question, leave it on the lecture page on my website or YouTube page
  - I would like to encourage discussion among those of you working through the material

Agenda

- Overview
  - Inherent characteristics of OD problem
- Dynamic System
  - Dynamic state estimation
    - Uniform gravitational field example
  - Observations
    - Non-linear functions of state variables
  - Linearization
  - State transition matrix
- Example Problem

Overview – Obtain Knowledge

- Orbit Determination is
  - The process by which we *obtain knowledge* of satellite motion
    - "The problem of determining the best estimate of the state of a spacecraft whose initial state is unknown, from observations influenced by random and systematic errors, using a mathematical model that is not exact, is … the problem of state estimation," or *orbit determination* [1]
- Let the astronomers focus on the heavenly bodies
  - We will focus on artificial (noncelestial) satellites

Overview – Characteristics

- Dynamic state estimation
  - Set of parameters required to predict future motion
  - Initially, just position and velocity vectors
  - Later, we will include dynamic and measurement model parameters
- Best estimate
  - Truth of observation and reported state

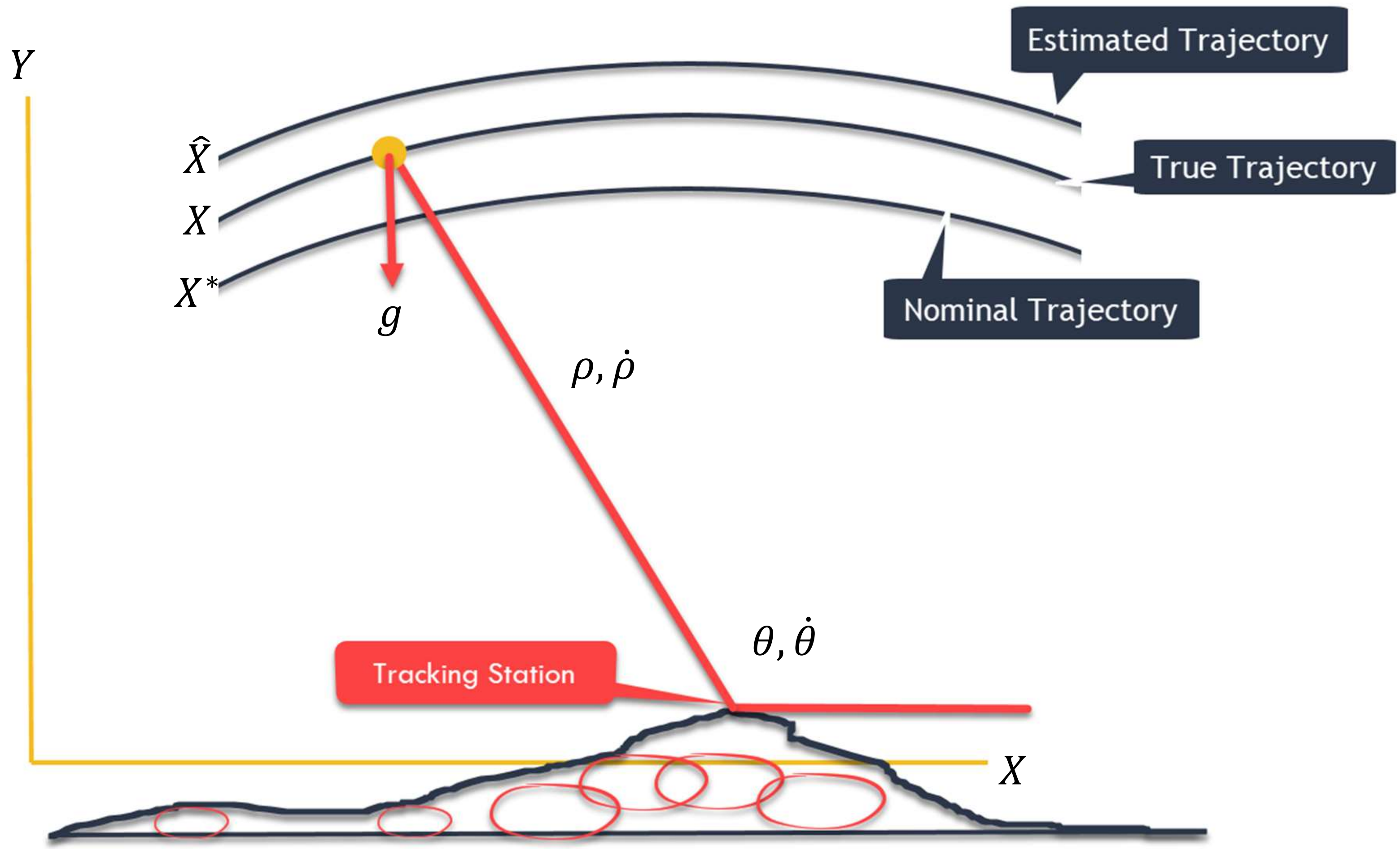
Overview – Estimates

- “Generating an ephemeris,” is predicting the state of a vehicle
  - An ephemeris is a table of position and velocity as function of time
- Predicted values differ from true values for 2 reasons
- 1. Inaccuracies in estimated state vector
  - Approximations in model and method of orbit propagation
  - Errors in observations
  - Errors in computational procedures
- 2. Errors in numerical integration
  - Caused by dynamical model and roundoff errors and truncation

Dynamic System – Estimates (1/2)

- Inaccuracies in estimate and errors in numerical integration
- Review probability, statistics and matrix theory
  - See Appendix A and B or notes on the appendices
- Will use a uniform gravitational field to illustrate OD process
  - True trajectory, X
  - Nominal trajectory, X^*
  - (Best) Estimate trajectory, $\hat{X}$
- X is our state vector
  - Observations
  - Location of Observations
  - Estimate method

Dynamic System – Estimates (2/2)



Dynamic System – Observing (1/3)

- Updating X is inherently linear
  - $X = A[\dot{X}]$
  - Ex: $X(t) = X_0 + \dot{X}_0 t$
- Cannot observe individual state components directly
- Observations are non-linear
  - Range, ρ, range-rate, $\dot{\rho}$, elevation, θ, elevation-rate, $\dot{\theta}$, etc.
  - $\rho(t) = \sqrt{(X(t) - X_s)^2 + (Y(t) - Y_s)^2}$
  - $\tan(\theta(t)) = (Y(t) - Y_s)/(X(t) - X_s)$
  - $\dot{\rho}(t) = \frac{1}{\rho}\left[(X(t) - X_s)\left(\dot{X}(t) - \dot{X}_s\right) + (Y(t) - Y_s)\left(\dot{Y}(t) - \dot{Y}_s\right)\right]$
  - $\dot{\theta}(t) = \frac{1}{\rho^2}\left[(X(t) - X_s)\left(\dot{Y}(t) - \dot{Y}_s\right) - \left(\dot{X}(t) - \dot{X}_s\right)(Y(t) - Y_s)\right]$

Dynamic System – Observing (2/3)

- Transforming observations into the best estimate

$$J(X_0) \equiv Y - G(X_0, t) = 0$$

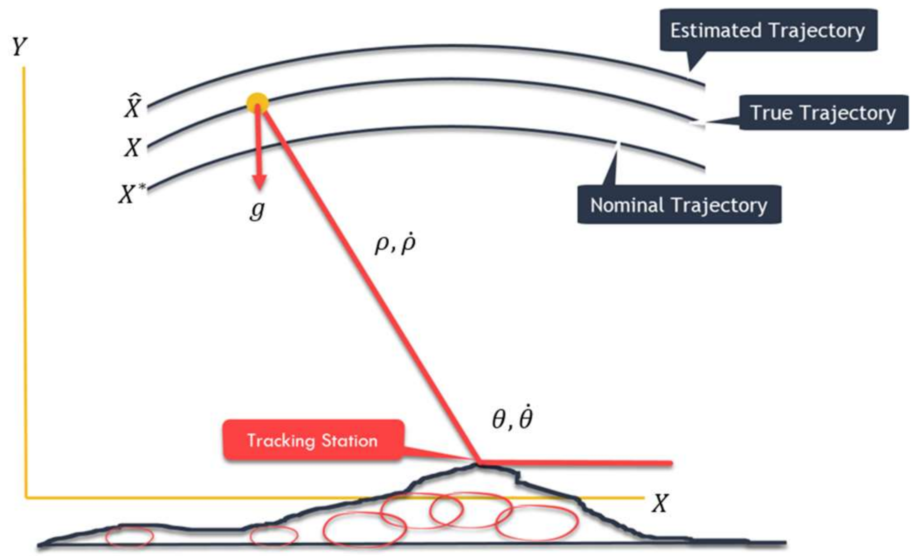
- Y is our observations and G is a 4x1 nonlinear vector from the equations in the previous slide
- Best estimate
  - Solve $J(X_0)$ using Newton-Raphson iteration
  - Iteration is repeated until $\left\| X_0^{n+1} - X_0^n \right\| \leq \varepsilon$
- Since equations are nonlinear, multiple solutions may exist

Dynamic System – Observing (3/3)

- Not all observation information may be available
  - Range and elevation but not range-rate nor elevation rate
- (ρ, θ) will be insufficient to determine X
  - Two observations at different times will be required
  - The set $(\rho_1, \theta_1, \rho_2, \theta_2)$ is sufficient
- Both approaches assume
  - Perfect knowledge of governing diff. eqns
  - Perfect observations
- In general many observations will be taken
  - This will decrease variance and eliminate random/systematic errors

Dynamic System – Linearization (1/4)

- Best estimate is obtained by linearizing the problem
  - Expand equations of motion
  - Expand observation-state relationship about a reference trajectory
- Deviations from reference are determined for best-fit
  - Minimize variance to yield best agreement with observations
  - Generally based on least-squares criterion



Dynamic System – Linearization (2/4)

- Using the flat-earth example in the previous slide
  - Assume errors in initial position, velocity, and in g values
  - $X = X^* + \delta X$ or the nominal plus the perturbed state
    - Perturbed means deviation (simplistically) in this case
- Write a state vector, subtracting the nominal
  - $\delta X = X - X^*, \delta X^T = [\delta X, \delta Y, \delta \dot{X}, \delta \dot{Y}, g]$
    - $\delta X = \delta X_0 + \delta \dot{X}_0 t$
    - $\delta Y = \delta Y_0 + \delta \dot{Y}_0 t - \delta g \left(\frac{t^2}{2}\right)$
    - $\delta \dot{X} = \delta \dot{X}_0$
    - $\delta \dot{Y} = \delta \dot{Y}_0 - \delta g t$
    - $\delta g = \delta g$

Dynamic System – Linearization (3/4)

- Observations becoming linear
  - Observations linearized by expanding in Taylor series about X^*
  - $\rho \cong \rho^* + \left[\frac{\delta\rho}{\delta X}\right]^* (X - X^*) + \left[\frac{\delta\rho}{\delta Y}\right]^* (Y - Y^*) + \varepsilon_\rho$
  - $\theta \cong \theta^* + \left[\frac{\delta\theta}{\delta X}\right]^* (X - X^*) + \left[\frac{\delta\theta}{\delta Y}\right]^* (Y - Y^*) + \varepsilon_\theta$
  - $\dot{\theta} \cong 0$
  - $\dot{\rho} \cong 0$
- Rewriting
  - $\delta\rho = \left[\frac{\delta\rho}{\delta X}\right]^* \delta X + \left[\frac{\delta\rho}{\delta Y}\right]^* \delta Y + \varepsilon_\rho$
  - $\delta\theta \cong \left[\frac{\delta\theta}{\delta X}\right]^* \delta X + \left[\frac{\delta\theta}{\delta Y}\right]^* \delta Y + \varepsilon_\theta$

Dynamic System – Linearization (4/4)

- Determining the deviation from the nominal trajectory
  - We have linearized the system
  - We can now use linear algebra to our advantage

$$y = \widetilde{H}x + \varepsilon$$

  - $y^T = [\delta\rho\ \delta\theta]$
  - $\widetilde{H}$ is the mapping vector of partial derivatives with respect to X^*
  - $x^T = \left[\delta X\ \delta Y\ \delta \dot{X}\ \delta \dot{Y}\ \delta g\right]$
  - $\varepsilon^T = \left[\varepsilon_\rho\ \varepsilon_\theta\right]$
- Think about it: How can you use this? Why is this valuable?

Dynamic System – State Transition Matrix (1/2)

- The state transition matrix (Φ) updates the deviation vector
  - $[\delta X] = \Phi(t, t_0)[\delta X_0]$
  - In other words, Φ maps deviations in the state vector from one time to another
- Classical orbit determination the mapping is exact
- General orbit determination
  - State equations are nonlinear
  - Φ is the linear term in a Taylor series expansion of $X(t)$ at t_0

Dynamic System – State Transition Matrix (2/2)

- Φ maps x_0 to $x(t)$
  - $y(t) = \left[\widetilde{H}(t)\Phi(\mathrm{t},\mathrm{t}_0)\right][\Phi(t,t_0)x_0] + \varepsilon$
- Given an arbitrary epoch, t_k
  - We can determine the best estimate of x_k using the deviations
  - H will be a $m \times n$ vector
    - n is the number of state variables
    - m is the number of observations
  - In general OD $m \geq n$ is always satisfied
  - In classical OD $m = n$
- We will cover how to use these extra observations later

Practice Problem: Uniform Gravitational Field (2D)

SIMPLE DYNAMIC SYSTEM

Simple Dynamic System – Problem Definition

Given perturbed initial conditions, use a Newton iteration scheme to recover the exact initial conditions or conditions used to produce the observations provided. Assume the ground station coordinates are correct.

- Write a computer program that computes $\rho(t_i)$
  - 2D uniform gravity field
- Compute observations from given initial conditions
- Iterate and solve for the correct initial conditions
- $X_0 = 1.5, Y_0 = 10.0, \dot{X}_0 = 2.2, \dot{Y}_0 = 0.5, g = 0.3, X_s = Y_s = 1.0$
- $\rho(t = 0, 1, 2, 3, 4) =$ $7.0, 8.00390597, 8.94427191, 9.801147892, 10.630145813$